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A uniformly valid solution for the hypersonic flow past blunted bodies

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The plane and axisymmetric hypersonic flow past blunted bodies is investigated as an inverse problem (shock shape given). The fluid may behave as a real gas in local thermodynamic equilibrium. Viscosity and heat conduction are neglected. An analytical solution uniformly valid in the whole flow field (from the stagnation region up to large distances from the body nose) is given. The solution is based on two main assumptions: (i) the density ratio ϵ across the shock is very small, (ii) the pressure at a point P of the disturbed flow field is *not* very small compared with the pressure immediately behind the shock in the intersection point of the shock surface with its normal through P. Terms $O(\epsilon)$ are neglected in comparison with 1, but it is not necessary for the shock layer to be thin. The change of velocity along streamlines is taken into account. In order to calculate the flow quantities one has to evaluate only two integrals (equations (49) and (53) together with the boundary values (5) and (10)). The application of the solution is illustrated and the accuracy is tested in some examples.

1. Introduction

Plane and axisymmetric hypersonic flows past blunt bodies have been investigated already by several authors. Two different problems have been examined. In the direct problem the shape of the body is given. In the inverse problem, however, the shape of the shock wave is given, and it is required to find the corresponding flow field and the shape of the body producing the shock.

The subsonic and transonic region of the flow field has been considered most extensively. This region can be treated separately, because a change of the body shape in the supersonic domain downstream of the limiting characteristic does not influence the flow in the subsonic region. For the following investigations the papers by Chester (1956) and Freeman (1956) are particularly important. These authors expanded the differential equations governing the flow of a perfect gas in powers of $(\gamma-1)/(\gamma+1)$ and M_∞^{-2}, where γ is the ratio of specific heats and M_∞ is the Mach number of the free stream. The first approximation for the pressure is identical with a formula given by Busemann (1933). The terms 'Newton–Busemann theory' or 'Newtonian theory plus centrifugal correction' are used for this kind of solution. The theory of Chester and Freeman gives exact solutions for $\gamma = 1$ and $M_\infty = \infty$. However, for all common gases γ is larger than 1. In this case the results of Chester and Freeman differ considerably from more accurate numerical or from experimental values, even for small values of the parameter $(\gamma-1)/(\gamma+1)$. Van Dyke (1958) pointed out that the series converge—if at all—very slowly. It may be mentioned that Cheng & Gaitatzes (1966) added a correction term to the first approximation of Chester and Freeman on the strength of qualitative considerations. Thus Cheng & Gaitatzes obtained more accurate results in the stagnation region.

Another difficulty arises at some distance downstream of the stagnation point. The theory of Chester and Freeman is not valid in regions where the pressure is a small fraction of its value at the nose. Freeman (1956) noticed a singularity, which appears in his solution for certain body shapes. (On a sphere, for instance, this singularity arises at a central half angle of 60°.) Attempts to continue the solution downstream of the singular point did not yield satisfactory results (see the discussion by Hayes & Probstein 1966, pp. 366–79).

Also the numerical methods, which have been developed for blunt bodies, are not useful in the supersonic region, but they can be continued by numerical methods of characteristics. We mention Inouye & Lomax (1962) and Honda (1965, 1966) as illustrative references for applications and results.

In a good number of papers the flow far downstream of the blunt nose of a slender body has been considered. The blast-wave analogy and the similarity solutions for power-law bodies have to be mentioned here. These theories fail in the subsonic region, where the shock slope is not small compared with 1. Moreover, there is another limitation of the applicability. Similarity solutions and blast-wave analogy are not valid within a layer near the body surface. This so-called entropy layer is characterized by large entropy and small density values and can occupy a considerable part of the space between shock wave and body surface. Sychev (1960) considered a paraboloidal shock shape and calculated numerically the influence of the entropy layer on the flow far downstream of the blunt nose of a slender body. Yakura (1962) used the method of matched asymptotic expansions to investigate the same problem. An 'outer' expansion describes the flow outside of the entropy layer, an 'inner' expansion is valid in the entropy layer and the two expansions are matched to give a single solution.

The methods, mentioned so far, are useful only in certain local regions, either in the subsonic and transonic region only, or in the supersonic region only, or only at large distances from the stagnation point. By 'joining' the individual solutions numerical difficulties arise, and from a theoretical point of view this procedure is not very satisfactory. An interesting attempt to find a solution valid in the whole flow field has been made by Maslen (1964). Maslen assumed, like Chester and Freeman, that the density downstream of the shock is much larger than in the free stream. But he further assumed that the velocity in an arbitrary point P of the shock layer is equal to the velocity immediately behind the shock wave in the intersection point of the shock surface with its normal through P. However, as Maslen himself noted, his method does not correspond to a rational theory and can be justified only by a comparison with other methods or experimental results. Moreover, in the stagnation region the assumption for the velocity is not even approximately satisfied.

Another way of describing the flow uniformly has been pointed out by Schneider

(1966). Whereas in the earlier paper an integral form of the energy equation has been used, in the present method we shall satisfy the energy equation correctly on each streamline. The aim of the following investigation of the hypersonic blunt-body problem is to find a solution, which is *uniformly valid in the whole flow field* (from the stagnation region up to large distances from the body nose). Furthermore, it will be easily seen that the solution can be applied to pointed bodies too.

2. Notation and fundamental equations

Consider the plane or axisymmetric hypersonic flow past a body. The shape of the bow shock wave is given, and it is required to find the corresponding flow field and the shape of the body producing the shock. Let the flow be inviscid and without heat conduction, and let the gas be in local thermodynamic equilibrium. The gas may have arbitrary thermal and caloric properties.

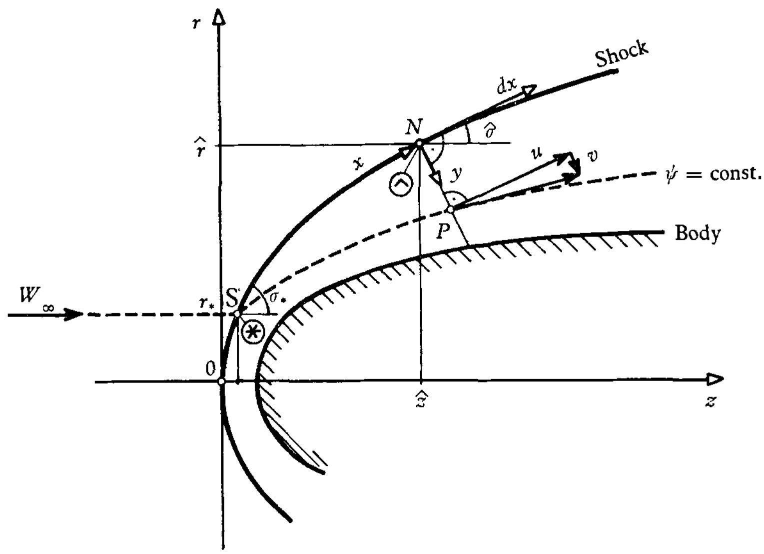

FIGURE 1. Co-ordinate systems and notation.

We introduce z and r as Cartesian co-ordinates for plane flow, and as cylindrical co-ordinates for axisymmetric flow. The z-axis is parallel to the direction of the free stream. It is advantageous to use additionally a shock-oriented co-ordinate system (x, y) of boundary-layer type (figure 1); x is the distance along the shock surface in the plane formed by the shock normal and the direction of the uniform stream, and y is the distance normal to the shock surface. From figure 1 one can draw the relations

$$z = \hat{z} + y \sin\hat{\sigma}, \tag{1a}$$

$$r = \hat{r} - y \cos\hat{\sigma}, \tag{1b}$$

$\hat{\sigma}$ is the shock inclination angle in the point $N(\hat{z}, \hat{r})$. Since the functions $\hat{z}(x)$ and $\hat{r}(x)$ are known for a given shock shape, the equations (1 a) and (1 b) may be used to calculate the co-ordinates z and r of a point P from its co-ordinates x and y.

The components of the velocity in the x- and y-directions are denoted by u and v respectively; ρ is the density, p the pressure, i the specific enthalpy and s the specific entropy. The curvature of the shock contour in the point N is $\hat{K}$, defined as positive if the centre of curvature lies at positive y ($\hat{K} = d\cos\hat{\sigma}/d\hat{r}$). The continuity equation can be written

$$\frac{\partial}{\partial x}(r^j\rho u) + \frac{\partial}{\partial y}[(1-\hat{K}y)\, r^j\rho v] = 0, \tag{2}$$

where $j = 0$ for plane flow, and $j = 1$ for axisymmetric flow. To satisfy the continuity equation (2), we introduce a stream function ψ by

$$\frac{\partial\psi}{\partial x} = (1-\hat{K}y)\, r^j\rho v; \tag{3a}$$

$$\frac{\partial\psi}{\partial y} = -r^j\rho u. \tag{3b}$$

With ψ and $\bar{x} = x$ as independent variables, the equations of energy, entropy and momentum are:

$$u^2+v^2+2i = u_*^2+v_*^2+2i_* = \text{const}; \tag{4}$$

$$\frac{\partial s}{\partial \bar{x}} = 0, \quad \text{or} \quad s = s_*(\psi); \tag{5}$$

$$u\frac{\partial u}{\partial \bar{x}} + v\frac{\partial v}{\partial \bar{x}} + \frac{1}{\rho}\frac{\partial p}{\partial \bar{x}} = 0; \tag{6}$$

$$(1-\hat{K}y)\left(1-j\frac{y}{\hat{r}}\cos\hat{\sigma}\right)\hat{r}^j\frac{\partial p}{\partial\psi} = \hat{K}u + \frac{\partial v}{\partial \bar{x}}. \tag{7}$$

Note that partial derivatives with respect to x are taken with y constant, but derivatives with respect to $\bar{x}$ are taken with ψ constant. From equations (3a) and (3b) we obtain the transformation equations

$$\frac{\partial y}{\partial \bar{x}} = (1-\hat{K}y)\frac{v}{u}; \tag{8a}$$

$$\frac{\partial y}{\partial \psi} = -\frac{1}{\{1-j(y/\hat{r})\cos\hat{\sigma}\}\hat{r}^j\rho u}. \tag{8b}$$

The equations (6), (7), (8a), (8b) and (4) or (5) provide a set of five equations for the five dependent variables (u, v, y and two thermodynamic quantities). As boundary conditions the flow quantities immediately behind the shock are given. It will be convenient to use two different notations for quantities behind the shock. Let two points N and S on the shock wave correspond to each point P of the disturbed flow field (figure 1). N is the point where the shock-normal ($\bar{x}$ = const) through P intersects the shock surface and S is the point where the streamline (ψ = const) through P crosses the shock wave. The flow quantities immediately behind the shock *in the point N are denoted by the superscript* ^, *and in the point S by the subscript* $_*$. We have already used this convention in equation (5). The analogous notation is used for the shock angle σ.

Velocity, density, pressure and specific enthalpy of the uniform stream are denoted by W_∞, ρ_∞, p_∞ and i_∞ respectively. In terms of the density ratio across the shock, defined by

$$\hat{\epsilon} = \rho_\infty/\hat{\rho}, \quad \epsilon_* = \rho_\infty/\rho_*, \tag{9}$$

the flow quantities immediately behind the shock are

$$\left.\begin{aligned} \hat{u} &= W_\infty \cos\hat{\sigma}, \\ \hat{v} &= W_\infty \hat{\epsilon} \sin\hat{\sigma}, \\ \hat{p} &= p_\infty + \rho_\infty W_\infty^2 (1-\hat{\epsilon}) \sin^2\hat{\sigma}, \\ \hat{i} &= i_\infty + \tfrac{1}{2} W_\infty^2 (1-\hat{\epsilon}^2) \sin^2\hat{\sigma}. \end{aligned}\right\} \tag{10}$$

Replacing $\hat{\sigma}$ and $\hat{\epsilon}$ by σ_* and ϵ_*, respectively, we get analogous equations for u_*, v_*, p_* and i_*.

The stream function ψ, defined by (3), contains a constant of integration. This constant is chosen such that $\psi = 0$ is the body stream line. Then ψ represents the mass flow between the streamline $\psi = \text{const}$ and the body surface, per unit depth for plane flows, and per unit azimuthal angle (in radians) for axisymmetric flows. Therefore the value $\hat{\psi}$ of the stream function at the shock in the point N (figure 1) is

$$\hat{\psi} = \rho_\infty W_\infty \frac{\hat{r}^{1+j}}{1+j}; \tag{11}$$

on the other hand, the co-ordinate r_* of the point S is connected with ψ by the relation

$$\psi = \rho_\infty W_\infty \frac{r_*^{1+j}}{1+j}. \tag{12}$$

3. Assumptions and approximations

For many real gases the density immediately behind a strong shock is much larger than in front of the shock (see, for instance, Lighthill 1957 or Hayes & Probstein 1966). Therefore we assume that

$$\hat{\epsilon} \ll 1 \quad \text{and} \quad \epsilon_* = O(\hat{\epsilon}). \tag{13}$$

As a second important assumption we shall use the relation

$$\hat{p}/p = O(1) \quad (\text{on } x = \text{const}, y > 0). \tag{14}$$

As is usual in fluid mechanics, the symbol $f(x) = O(g(x))$ means throughout this paper that $|f(x)|$ is not very large in comparison with $|g(x)|$. Therefore equation (14) states that the pressure in a point P of the disturbed flow field is not very small compared with the pressure immediately behind the shock in the intersection point of the shock surface with its normal through P. It should be noted that an analogous relation for p_*/p is *not* assumed. Thus the pressure ratio p_*/p may be very large, so that a strong expansion on a streamline is permitted. (Strong expansions on streamlines occur in flows past slender bodies with nose blunting.) The assumption (14) may be applied to many body shapes of practical interest, and it may be applied also far downstream of the body nose.

(For blunted flat plates and blunted cylinders, as examples, $\hat{p}/p$ is about 2 according to the blast-wave solutions for the flow far downstream of the nose.)

We now consider the momentum equation (7). With the assumptions (13) and (14) we shall obtain (32). Equations like (32) are used in the theory of thin shock layers, but we shall show that the equation is valid more generally. Integrating formally equation (7) with the boundary condition $p = \hat{p}$ at $\psi = \hat{\psi}$ we obtain

$$\hat{r}^j(\hat{p}-p)[1+O(J_1)] = \hat{K}\int_\psi^{\hat{\psi}} u\,d\psi + J_2, \tag{15a}$$

where
$$J_1 = \left(\hat{K}+j\frac{\cos\hat{\sigma}}{\hat{r}}\right)\frac{1}{\hat{\psi}-\psi}\int_\psi^{\hat{\psi}} y\,d\psi; \tag{15b}$$

$$J_2 = \int_\psi^{\hat{\psi}} \frac{\partial v}{\partial \bar{x}}\,d\psi. \tag{15c}$$

Since y is a decreasing monotonic function of ψ, J_1 is of order

$$J_1 = \left(\hat{K}+j\frac{\cos\hat{\sigma}}{\hat{r}}\right)O(\delta_{\text{eff}}), \tag{16}$$

where the 'effective layer thickness' δ_{eff} is defined by

$$\delta_{\text{eff}} = \frac{1}{\hat{\psi}}\int_0^{\hat{\psi}} y\,d\psi. \tag{17}$$

Its physical meaning will be explained in the next section.

In order to calculate $O(\delta_{\text{eff}})$ we need estimations of $1/u$ and $1/\rho$. Since the entropy increases with increasing shock angle σ, we have $\hat{s} = O(s_*)$. With (14) it follows that $\hat{\imath} = O(i)$ and from the energy equation (4) we conclude that

$$u = O(\hat{u}). \tag{18}$$

On the other hand, $1/u$ is $O(1/u_*)$, because the velocity increases along streamlines with falling pressure. Noticing (10) we see that $1/u = O(1/\hat{u})$, except perhaps the region $0 \leqslant \psi \ll \hat{\psi}$, where $\cos\sigma_*$ can be very small compared with $\cos\hat{\sigma}$. To obtain approximations valid in this region, too, we write

$$\frac{1}{u} = O\left(\frac{1}{u_b+[\hat{u}-u_b]\,\psi/\hat{\psi}}\right), \tag{19}$$

where the subscript b denotes the value on the body surface. In order to find an analogous equation for ρ, we replace i and p in

$$\frac{\partial\rho}{\partial\psi} = \left(\frac{\partial\rho}{\partial p}\right)_i\frac{\partial p}{\partial\psi}+\left(\frac{\partial\rho}{\partial i}\right)_p\frac{\partial i}{\partial\psi} \tag{20}$$

by means of (4) and (7). As the thermodynamic functions $(\partial\rho/\partial p)_i$ and $(\partial\rho/\partial i)_p$ for a gas do not change their orders of magnitude if p and i keep their orders of magnitude, and since v is of the order of u at most, except in the stagnation region, it follows from (20) with the aid of (15a) that the order of magnitude of

$\partial\rho/\partial\psi$ does not depend on ψ, except perhaps the region near the body surface, where u can be much smaller than $\hat{u}$. Thus we have

$$\frac{1}{\rho} = \left(\frac{1}{\rho_b + [\hat{\rho} - \rho_b]\,\psi/\hat{\psi}}\right). \tag{21}$$

As is well known, ρ is almost constant in the region near the stagnation point (see Lighthill 1957; Hayes & Probstein 1966). One recognizes immediately that (21) is valid in the stagnation region too.

With the help of (19), (21) and (11), equation (8*b*) gives

$$y = O\left(\frac{\hat{\epsilon}\hat{r}}{\cos\hat{\sigma}}\frac{1}{u_b/\hat{u} - \rho_b/\hat{\rho}}\ln\frac{u_b/\hat{u} + [1 - u_b/\hat{u}]\,\psi/\hat{\psi}}{\rho_b/\hat{\rho} + [1 - \rho_b/\hat{\rho}]\,\psi/\hat{\psi}}\right), \tag{22}$$

and it follows from (17) that

$$\delta_{\text{eff}}\frac{\cos\hat{\sigma}}{\hat{r}} = O(\hat{\epsilon}F), \tag{23}$$

where
$$F = \frac{1}{u_b/\hat{u} - \rho_b/\hat{\rho}}\left[\frac{\rho_b/\hat{\rho}}{1 - \rho_b/\hat{\rho}}\ln\frac{\rho_b}{\hat{\rho}} - \frac{u_b/\hat{u}}{1 - u_b/\hat{u}}\ln\frac{u_b}{\hat{u}}\right]. \tag{24}$$

The function F is $O(1)$ for any $u_b/\hat{u}$ and $\rho_b/\hat{\rho}$ lying in the physically possible range $u_b/\hat{u} = O(1)$ and $\rho_b/\hat{\rho} = O(1)$, where $u_b/\hat{u}$ and $\rho_b/\hat{\rho}$ cannot be very small simultaneously, because an expansion into vacuum leads to very large velocities. Equation (23) then becomes

$$\delta_{\text{eff}}\frac{\cos\hat{\sigma}}{\hat{r}} = O(\hat{\epsilon}) \ll 1. \tag{25}$$

The physical meaning of this simple relation will be explained in the next chapter.

We return now to the integrals J_1 and J_2. Assuming $\partial v/\partial\bar{x}$ to be continuous in the closed interval $[\psi, \hat{\psi}]$, integration and differentiation may be exchanged in (15*c*) and the following estimate is obtained with (8*a*) and (18):

$$J_2 = O\left(\frac{d}{d\bar{x}}\left[\hat{u}\frac{d(\hat{\psi}\delta_{\text{eff}})}{d\bar{x}}\right]\right) + O\left(\hat{v}\frac{d\hat{\psi}}{d\bar{x}}\right). \tag{26}$$

We note that continuity of $\partial v/\partial\bar{x}$ in the interval $[\psi, \hat{\psi}]$ requires smooth streamlines in $[\psi, \hat{\psi}]$; if the results should be correct up to the body surface, also the body streamline has to be smooth. After substitution from (25), $d\hat{\epsilon}/d\bar{x}$ appears in equation (26). From

$$\hat{\epsilon} = \hat{\epsilon}(\rho_\infty, i_\infty, W_\infty\sin\hat{\sigma})$$

we obtain
$$\frac{d\hat{\epsilon}}{d\bar{x}} = \frac{\partial\hat{\epsilon}}{\partial(W_\infty\sin\hat{\sigma})}\frac{d(W_\infty\sin\hat{\sigma})}{d\hat{\sigma}}\frac{d\hat{\sigma}}{d\bar{x}} = O(\hat{K}\hat{\epsilon}\cot\hat{\sigma}). \tag{27}$$

Equations (11), (26) and (27) then combine to give

$$J_2 = \rho_\infty W_\infty^2\hat{\epsilon}\hat{r}^j\sin^2\hat{\sigma}\left[O(1) + O\left(\frac{\hat{K}\hat{r}}{\sin^2\hat{\sigma}\cos\hat{\sigma}}\right) + O\left(\frac{\hat{K}^2\hat{r}^2}{\sin^4\hat{\sigma}\cos^2\hat{\sigma}}\right) + O\left(\frac{(d\hat{K}/d\hat{r})\,\hat{r}^2}{\sin^2\hat{\sigma}\cos\hat{\sigma}}\right)\right]. \tag{28}$$

With the help of (15a) and (18) it may be concluded from the assumption (14) that

$$\frac{\hat{K}\hat{r}}{\sin^2\hat{\sigma}\cos\hat{\sigma}} = O(1). \tag{29}$$

Thus equation (28) becomes

$$J_2 = \rho_\infty W_\infty^2 \hat{r}^j \sin^2\hat{\sigma} O(\hat{\epsilon}) \tag{30}$$

and equation (16) with (25) may be rewritten

$$J_1 = O(\hat{\epsilon}). \tag{31}$$

The relations (30) and (31) can now be substituted into equation (17). Noticing the equations (10), (13) and (14) and neglecting terms $O(\hat{\epsilon})$ in comparison with 1, we obtain

$$p = \hat{p} - \frac{\hat{K}}{\hat{r}^j}\int_{\psi}^{\hat{\psi}} u\,d\psi. \tag{32}$$

This simplified form of the momentum equation (7) will serve as basis for solving the problem. Equations analogous to (32) have been used already in the theory of thin shock layers. However, we have rederived this basic equation under assumptions which do not restrict the range of applicability to thin shock layers. We shall discuss this point in the next chapter.

4. Remarks upon the physical meaning of the approximations

A comparison of (32) with the exact momentum equation (7) shows that the following approximations are valid within the assumptions used. (i) The pressure gradient normal to the shock wave is approximately equal to the pressure gradient normal to the streamlines. This pressure gradient is due to centrifugal forces. (ii) In order to calculate the pressure—but only for the pressure—the streamline curvature may be taken equal to the shock curvature.

The mass flow (per unit depth for plane flow, and per unit azimuthal angle for axisymmetric flow) between the streamline ψ and a neighbouring streamline $\psi + d\psi$ is $d\psi$. Since y is the distance from the shock wave, the quantity δ_{eff} defined by (17) may be interpreted as distance of the 'mass flow centre' from the shock surface. Consequently, the number $(\delta_{\text{eff}}/\hat{r})\cos\hat{\sigma}$ provides a measure of the concentration of mass flow at the shock. Thus equation (25), which played an important role in the previous chapter, may be interpreted as follows: almost the whole mass flows very closely to the shock; or in other words, most of the streamlines drawn for constant differences of mass flow lie near the shock surface. But not all streamlines must lie near the shock. Especially the distance of the body surface from the shock need not to be small in comparison with $\hat{r}/\cos\hat{\sigma}$. Therefore it is *not* necessary to have a thin shock layer in order to apply equation (32) and the solution given in the next chapter. This fact is essential for the applicability of the theory to blunted slender bodies.

In equation (32) not only the pressure p but also the velocity component u is unknown. The theory of thin shock layers, as developed for blunt bodies by Chester (1956) and Freeman (1956), yields within the first approximation, that u

is constant on streamlines: $u = u_*(\psi)$. This result, however, is not valid in regions where the pressure is a small fraction of its value at the nose (see Chester 1956 and Freeman 1956). Therefore we shall follow another approach.

5. A uniformly valid solution

We proceed from the momentum equation (6), which we may rewrite

$$\frac{1}{2}\frac{\partial(u^2+v^2)}{\partial \bar{x}}+\frac{p}{\rho}\frac{\partial \ln p}{\partial \bar{x}} = 0. \tag{33}$$

The ratio of the velocity components can be estimated from equation (10) and figure 1:

$$\frac{v^2}{u^2} = O(\hat{\epsilon}^2 \tan^2 \hat{\sigma})+O\left(\frac{v_b^2}{u_b^2}\right). \tag{34}$$

Combining (8*a*) and (22) we obtain

$$\frac{v_b}{u_b} = O\left(\hat{\epsilon}\sin\hat{\sigma}\ln\frac{\rho_b}{\hat{\rho}}\right), \tag{35}$$

and with (10) and (14) we have

$$\frac{v_b}{u_b} = O\left(\hat{\epsilon}\left[\frac{p_b}{\rho_\infty W_\infty^2}\right]^{\frac{1}{2}}\ln\frac{\rho_b}{\hat{\rho}}\right). \tag{36}$$

Downstream of the shock the entropy does not change on streamlines, so that we may use the effective isentropic exponent

$$\gamma_e = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s = \frac{\rho}{p}\left(\frac{\partial p}{\partial \rho}\right)_s, \tag{37}$$

which is not smaller than 1 for any known gas. Thus on a streamline the relative pressure change is not smaller than the relative density change, so that we have for

$$d\rho \leqslant 0: \quad \frac{p}{\rho} = O\left(\frac{p_*}{\rho_*}\right) = O(\epsilon_* W_\infty^2 \sin^2\sigma_*). \tag{38}$$

For $d\rho > 0$, on the other hand, we note that $p/\rho_\infty W_\infty^2$ cannot be very large. Thus we may write for both $d\rho \leqslant 0$ and $d\rho > 0$

$$\frac{p}{\rho} = W_\infty^2\, O(\epsilon_*). \tag{39}$$

This equation is also valid on the body streamline, and equation (36) becomes

$$\frac{v_b}{u_b} = O\left(\left[\hat{\epsilon}\epsilon_*\frac{\rho_b}{\hat{\rho}}\right]^{\frac{1}{2}}\ln\frac{\rho_b}{\hat{\rho}}\right). \tag{40}$$

Because of the assumption (13) it follows that

$$\frac{v_b}{u_b} = O(\hat{\epsilon}) \tag{41}$$

for all values of $\rho_b/\hat{\rho}$ coming into question ($\rho_b/\hat{\rho} = O(1)$). If the stagnation region, defined by $\tan^2\hat{\sigma} \gg 1$ is excluded *temporarily* from the consideration, the equations (34) and (41) show that v^2 may be neglected in comparison with u^2 within the momentum equation (33). Integrating (33) and considering the isentropy on streamline, we obtain

$$u^2 - u_*^2 + 2\int_{\ln p_*}^{\ln p} \left(\frac{p}{\rho}\right)_{s=s_*} d(\ln p) = 0. \tag{42}$$

For the first term in this equation, we have the relation

$$W_\infty^2 = O(u^2) \tag{43}$$

if the stagnation region and the region characterized by $u \ll \hat{u}$ are excluded temporarily again (see the discussion subsequent to (18)). The magnitude of the third term in (42) may be evaluated from (39) as

$$\int_{\ln p_*}^{\ln p} \left(\frac{p}{\rho}\right)_{s=s_*} d(\ln p) = W_\infty^2\, O\left(\epsilon_* \ln \frac{p_*}{p}\right). \tag{44}$$

This term, though $\epsilon_* \ll 1$ has been assumed, is not to be neglected in equation (42), if it is required to find a uniformly valid solution. For, the gas may expand very much on streamlines coming out from the stagnation region, so that $\ln(p_*/p)$ can become very large. (Note that with neglecting the integral term, the solution $u = u_*(\psi)$ derived by Chester and Freeman is obtained immediately.) Since p is still unknown, the question arises how u may be calculated from (42) without neglecting the integral term. We can overcome this crucial difficulty as follows.

The integral in (42) is divided into two parts to give

$$u^2 - u_*^2 + 2\int_{\ln p_*}^{\ln \hat{p}} \left(\frac{p}{\rho}\right)_{s=s_*} d(\ln p) + 2\int_{\ln \hat{p}}^{\ln p} \left(\frac{p}{\rho}\right)_{s=s_*} d(\ln p) = 0. \tag{45}$$

The first integration extends from the lower limit $\ln p_*$ to the upper limit $\ln \hat{p}$. These limits are given by the boundary conditions at the shock. The second integration extends from $\ln \hat{p}$ to $\ln p$, which is unknown. But the assumption (14) with equation (32) allows us to write

$$\int_{\ln \hat{p}}^{\ln p} \left(\frac{p}{\rho}\right)_{s=s_*} d(\ln p) = W_\infty^2\, O\left(\epsilon_* \ln \frac{\hat{p}}{p}\right) = W_\infty^2\, O(\epsilon_*). \tag{46}$$

This equation together with (43) shows that we can neglect the second integral in (45) (also for strong expansions on streamlines). Then (45) becomes

$$u_{(p)}^2 = u_*^2 - 2\int_{p_*}^{\hat{p}} \left(\frac{1}{\rho}\right)_{s=s_*} dp + \dots \tag{47}$$

or (with $di = dp/\rho$ for $ds = 0$)

$$u_{(p)}^2 = u_*^2 + 2[i_* - i(\hat{p}, s_*)] + \dots, \tag{48}$$

where $i = i(p,s)$ is an equation of state of the gas. The subscript (p) denotes that this approximation is used only to calculate the pressure p. The reasons for this will appear later in connexion with equation (51). Now (48) may be substituted into (32). We obtain the result

$$p = \hat{p} - \frac{\hat{K}}{\hat{r}^j}\int_{\psi}^{\hat{\psi}} (u_*^2 + 2[i_* - i(\hat{p}, s_*)])^{\frac{1}{2}}\, d\psi. \tag{49}$$

On the right-hand side of this equation there are only quantities given by the boundary conditions at the shock or by the state equation of the gas.

Now we consider the regions we had excluded in deriving (49). The part of the flow field where $u \ll \hat{u}$ yields a negligible contribution to the integral in (32) or (49), since here the integrand is very small and, furthermore, this region occupies only a small part of the whole integration interval. In the stagnation region, however, the density is approximately constant (see Lighthill 1957; Whitham 1957; Hayes & Probstein 1966), and from Bernoulli's equation it follows that

$$\frac{p-\hat{p}}{\hat{p}} = O(\epsilon_*). \tag{50}$$

On the other hand, we obtain from (49) for $\hat{\psi} \to 0$ that $p = \hat{p}$ in the stagnation point. We recognize that the terms neglected in (49) are $O(\hat{\epsilon})$ within the whole flow field including the stagnation region.

With $s = s_*(\psi)$ and p evaluated from (49), the thermodynamic state is now known in the streamline co-ordinate system $(\bar{x}, \psi)$. In order to determine the location of a point in space, we need the distance y from the shock surface as a function of $\bar{x}$ and ψ. After separating the variables y and ψ in (8*b*), we may solve this equation without any approximation. The solution is

$$y\left(1-\frac{j\cos\hat{\sigma}}{2\hat{r}}\,y\right) = \frac{1}{\hat{r}^j}\int_{\psi}^{\hat{\psi}} \frac{d\psi}{\rho u}. \tag{51}$$

The density ρ can be calculated from the state equation $\rho = \rho(p, s)$. For the velocity component u the approximation (48) has already been given, but we do not use this equation here, because it is not valid in the stagnation region. Although it has been shown just now that equation (49) containing (48) is, nevertheless, consistent within the stagnation region, analogous conclusions cannot be drawn for (51). But since $p(\bar{x}, \psi)$ is already known, we can proceed now directly from the energy equation (4). As has been shown above, v^2 can be neglected in comparison with u^2 outside of the stagnation region. Hayes & Probstein (1966, pp. 235–7 and 247–9) pointed out that this leads to errors, which are $O(\hat{\epsilon})$ within the stagnation region too. Thus from the energy equation (4) it follows that

$$u^2 = u_*^2 + 2[i_* - i(p, s_*)] + \ldots, \tag{52}$$

and the integral in (51) becomes

$$Y = \int_{\psi}^{\hat{\psi}} \frac{d\psi}{\rho(p, s_*)\{u_*^2 + 2[i_* - i(p, s_*)]\}^{\frac{1}{2}}}. \tag{53a}$$

Solving the left-hand side of (51) with respect to y, one has to distinguish between plane ($j = 0$) and axisymmetric ($j = 1$) flows. This gives

$$\left.\begin{aligned} &\text{for}\quad j = y\colon\quad 0 = Y;\\ &\text{for}\quad j = y\colon\quad 1 = \frac{\hat{r}}{\cos\hat{\sigma}}\left[1-\left(1-\frac{2Y\cos\hat{\sigma}}{\hat{r}^2}\right)^{\frac{1}{2}}\right].\end{aligned}\right\} \tag{53b}$$

With (53*a*) and (53*b*) the solution (49) depending on $\bar{x}$ and ψ is obtained in the co-ordinate system x, y. Finally, the equations (1*a*) and (1*b*) give the solution in the convenient co-ordinates z and r.

On the body surface we have $\psi = 0$. Therefore we obtain the pressure p_b on the body surface and the shock layer thickness δ by replacing the lower limit in (49) and (53 a) by zero. Hence we have

$$\psi = 0: \quad \left.\begin{aligned} p &= p_b(x); \\ y &= \delta(x). \end{aligned}\right\} \tag{54}$$

Perfect gas

For a perfect gas with constant specific heats the density ratio across the shock is given by the well-known relation

$$\epsilon_* = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_\infty^2 \sin^2\sigma_*}, \tag{55}$$

where γ is the ratio of specific heats. By use of the shock conditions (10), together with (55), the equations (49) and (53 a) become

$$p = \hat{p} - \frac{W_\infty \hat{K}}{\hat{r}^j} \int_\psi^{\hat{\psi}} \left\{ \cos^2\sigma_* + \left[\frac{2}{(\gamma-1)M_\infty^2} + \sin^2\sigma_*\right]\left[1 - \left(\frac{\sin^2\hat{\sigma}}{\sin^2\sigma_*}\right)^{(\gamma-1)/\gamma}\right]\right\}^{\frac{1}{2}} d\psi; \tag{56}$$

$$Y = \frac{1}{\rho_\infty W_\infty} \int_\psi^{\hat{\psi}} \frac{\epsilon_*(\hat{p}\sin^2\sigma_*/p\sin^2\hat{\sigma})^{1/\gamma}\, d\psi}{\left\{\cos^2\sigma_* + \left[\frac{2}{(\gamma-1)M_\infty^2} + \sin^2\sigma_*\right]\left[1 - \left(\frac{p\sin^2\hat{\sigma}}{\hat{p}\sin^2\sigma_*}\right)^{(\gamma-1)/\gamma}\right]\right\}^{\frac{1}{2}}}. \tag{57}$$

These equations show very clearly the difference existing between the present method and the Newtonian theory (with centrifugal correction). By putting $M_\infty = \infty$ and $\gamma = 1$ in (56) we obtain exactly the Busemann formula (in which velocity changes on streamlines are neglected). This connexion is consistent with the fact, that for $\gamma = 1$ an isentropic expansion is isothermal, and the velocity remains constant according to the energy equation. If, however, γ is larger than 1, say $\gamma = (1+\epsilon)/(1-\epsilon)$, where $\epsilon > 0$, an essential change of the velocity magnitude can result even for arbitrarily small ϵ, provided that the pressure change is sufficiently large. This is taken into account in (56) and (57) by the terms, which appear in the radicands additionally to $\cos^2\sigma_*$. The importance of these terms for expansions in powers of $(\gamma-1)/(\gamma+1)$ has been explained by Schneider (1966). It appeared that the additional terms are essential as well far downstream as in the stagnation region. Also they are necessary to avoid singularities in the solution. The limiting process $\gamma \to 1$ for strong blast waves has been studied by Freeman (1960) and Brocher (1960). Mirels (1962) considered the same limiting process for the steady hypersonic flow past slender bodies and drew analogous conclusions.

It may be interesting that the uniformly valid solution (49) can be obtained somewhat more simply for a perfect gas than for a real gas. We proceed from the energy equation (4) and neglect v^2 and v_*^2 according to (34) and (41). The enthalpy for a perfect gas with constant specific heats is

$$i = \frac{\gamma-1}{\gamma} p^{(\gamma-1)/\gamma} \exp(s/c_p)$$

and can be rewritten as

$$i(p, s_*) = i(\hat{p}, s_*) \frac{i(p, s_*)}{i(\hat{p}, s_*)} = i(\hat{p}, s_*) \left(\frac{p}{\hat{p}}\right)^{(\gamma-1)/\gamma}. \tag{58}$$

Expanding in powers of $(\gamma - 1)/\gamma$ and using the assumption (14) we obtain

$$i(p, s_*) = i(\hat{p}, s_*) \left[1 + O\left(\frac{\gamma - 1}{\gamma}\right)\right]. \tag{59}$$

$(\gamma - 1)/\gamma$ is $O(\epsilon_*)$ according to (55). Thus the energy equation (4) reduces to (48), and (32) yields the result (49).

6. Examples and discussion of the results

Most frequently in the literature, the paraboloidal shock wave ($j = 1, \hat{r}^2 = 2\hat{z}$) in a stream of perfect gas ($\gamma = 1{\cdot}4$) at infinite Mach number has been treated. For this test problem, results of the present method are plotted in the figures 2–5. Comparisons are drawn with numerical results reported by Van Dyke & Gordon (1959) and Van Dyke (1966) for the subsonic and transonic region of the flow

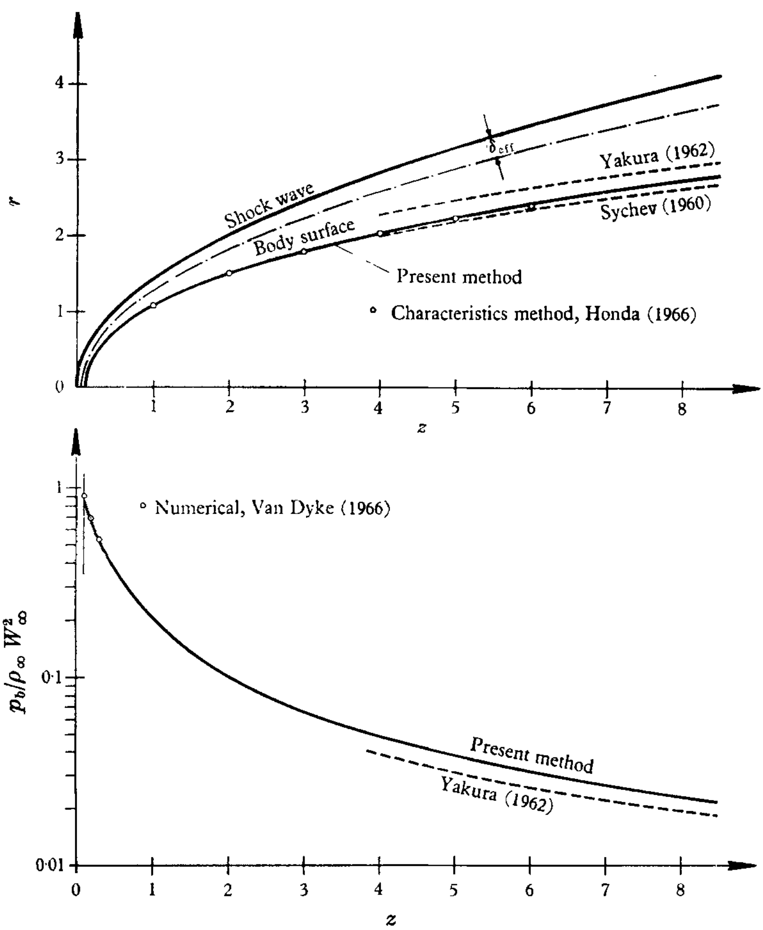

FIGURE 2. Body shape and pressure at body surface for a paraboloidal shock wave ($j = 1$, $\hat{r}^2 = 2\hat{z}$).

field, with the method of characteristics (Honda 1966) in the near supersonic region, and with asymptotic solutions derived by Sychev (1960) and Yakura (1962) for regions far downstream of the body nose. The agreement with the numerical methods is quite good, although $\hat{\epsilon}$ is already relatively large in comparison with the assumption (13). ($\hat{\epsilon} = (\gamma - 1)/(\gamma + 1) = \frac{1}{6}$.) In figure 2 the effective layer thickness δ_{eff} defined by (17) is also given. This quantity and the density profile in

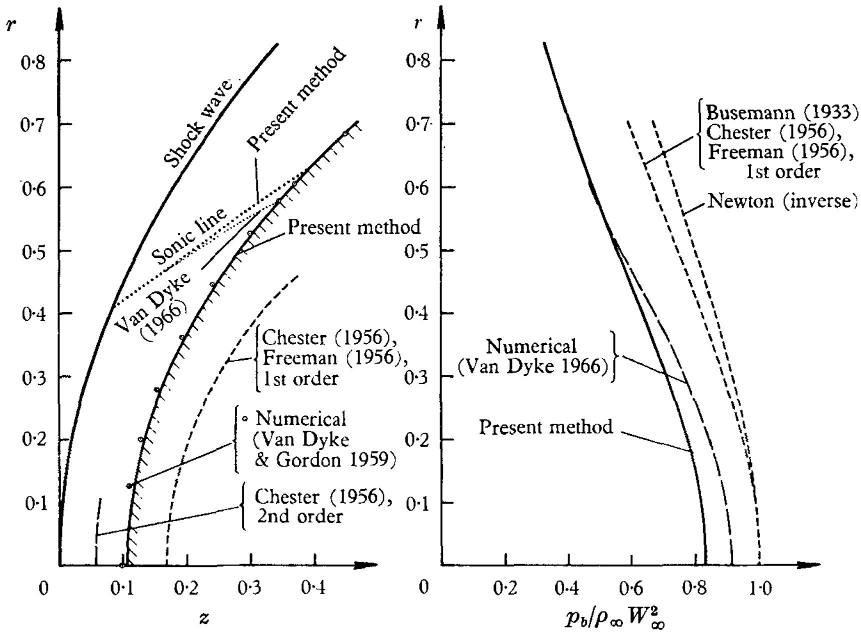

FIGURE 3. Body shape and pressure at body surface for a paraboloidal shock wave ($j = 1$, $\hat{r}^2 = 2\hat{z}$). Perfect gas, $\gamma = 1{\cdot}4$, $M_\infty = \infty$. Subsonic and transonic region.

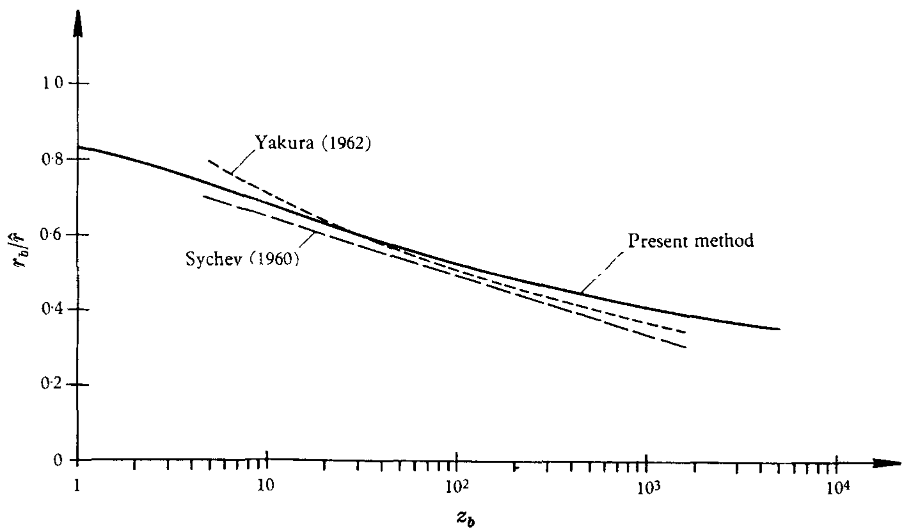

FIGURE 4. Ratio of body radius to shock radius in sections normal to a paraboloidal shock wave ($j = 1$, $\hat{r}^2 = 2\hat{z}$). Perfect gas, $\gamma = 1{\cdot}4$, $M_\infty = \infty$. Large distance from the body nose.

figure 5 show clearly the concentration of the mass flow at the shock wave. Likewise remarkable is the entropy profile, from which we may notice the entropy layer. However, a definite boundary between the entropy layer and the outer shock layer cannot be seen. In figure 5 it occurs that the pressure distribution according to Yakura's solution differs somewhat from the distribution calculated

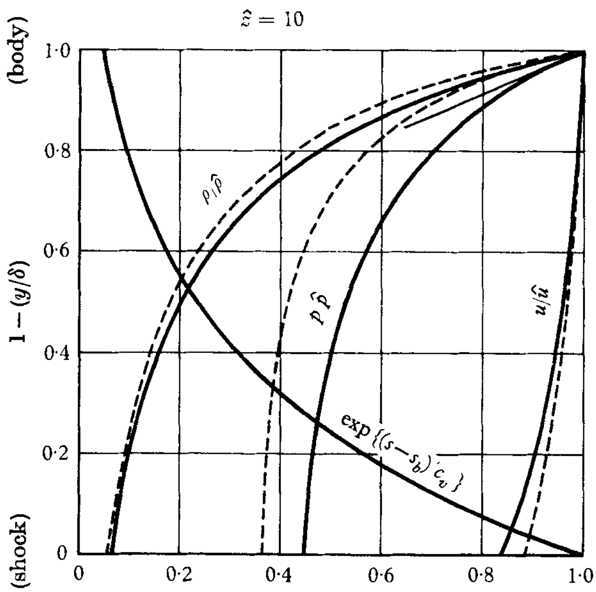

FIGURE 5. Distribution of flow quantities on a normal to a paraboloidal shock wave $j = 1$, $\hat{r}^2 = 2\hat{z}$. Perfect gas, $\gamma = 1{\cdot}4$, $M_\infty = \infty$. ——, present method; – – –, Yakura (1962).

by the present method. In order to test the two methods on their accuracy, we compare the pressure gradients normal to the shock surface. From the inclination and the curvature of the shock the following exact result valid for $M_\infty = \infty$ can be deduced:

$$\left.\frac{\partial(p/\hat{p})}{\partial y}\right|_{y=0} = -2\hat{K}\left[\frac{2\gamma-1}{\gamma-1}\cot^2\hat{\sigma} - \frac{\gamma}{\gamma+1}\left(1+\frac{j\cos\hat{\sigma}}{\hat{K}\hat{r}}\right)\right]. \tag{60}$$

By using (49), (53*a*), (53*b*) and (10) we obtain

$$\left.\frac{\partial(p/\hat{p})}{\partial y}\right|_{y=0} = -\frac{\hat{K}\cot^2\hat{\sigma}}{(1-\hat{\epsilon})\hat{\epsilon}} \tag{61}$$

according to the present theory. For the example of a paraboloidal shock wave and $\gamma = 1{\cdot}4$ we draw from Yakura's (1962, figure 14) paper a value of about $-3{\cdot}6$ as pressure gradient at $\hat{z} = 10$. We compare this value with the results obtained from (60) and (61):

$$\left.\frac{\partial(p/\hat{p})}{\partial y}\right|_{\substack{y=0\\ \hat{z}=10}} \begin{cases} = -1{\cdot}60 & \text{(exact);} \\ = -1{\cdot}50 & \text{(present method);} \\ \approx -3{\cdot}6 & \text{(Yakura 1962).} \end{cases}$$

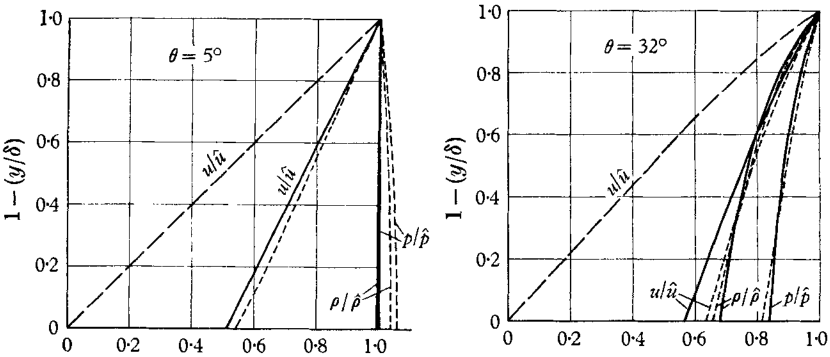

FIGURE 6. Distribution of flow quantities on two normals to a spherical shock wave. Perfect gas, $\gamma = 1{\cdot}2$, $M_\infty = 10$. The central half angle is equal to 5° and 32° respectively. ——, present method; – – –, Zlotnick & Newman (see Hayes & Probstein 1966); — — —, Freeman (1956).

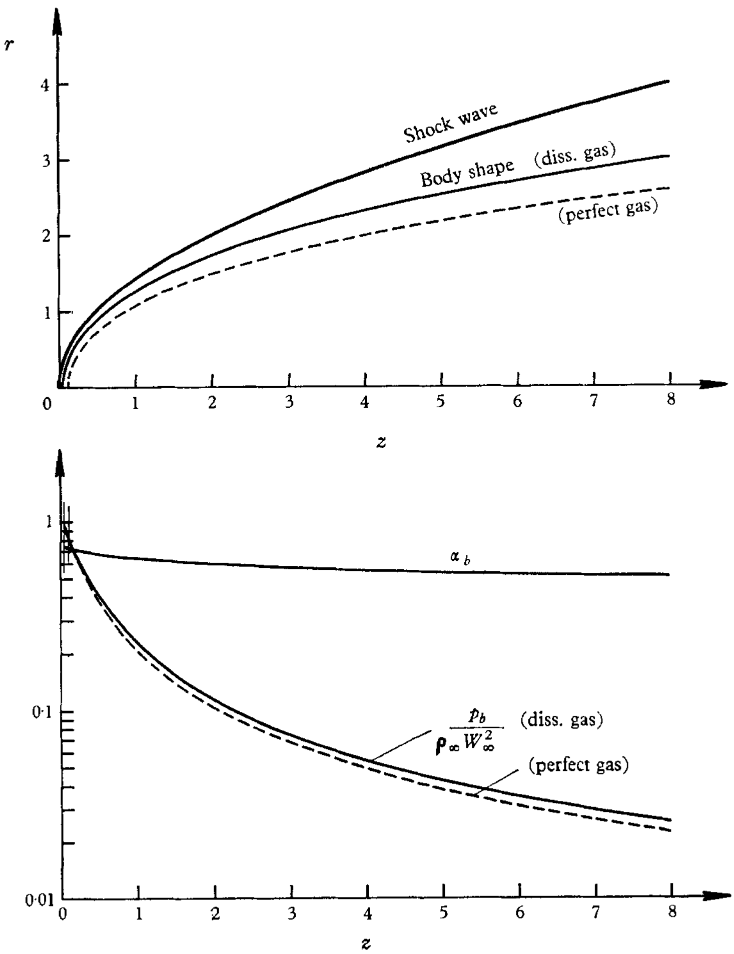

FIGURE 7. Body shape and pressure at body surface for a paraboloidal shock wave ($j = 1$, $\hat{r}^2 = 2\hat{z}$). Ideal dissociating gas, $W_\infty^2/RT_d = 2$, $\rho_\infty/\rho_d = 10^{-9}$, $T_\infty/T_d = 5\times10^{-3}$ ($M_\infty = 17{\cdot}32$). Perfect gas, $\gamma = 1{\cdot}4$, $M_\infty = 17{\cdot}32$.

Hence we see that in this test case the results of the present method are more accurate than those of Yakura's method.

For the neighbourhood of the stagnation point, results of the present method are compared in figure 6 with results numerically evaluated by Zlotnick & Newman (see Hayes & Probstein 1966, p. 460).

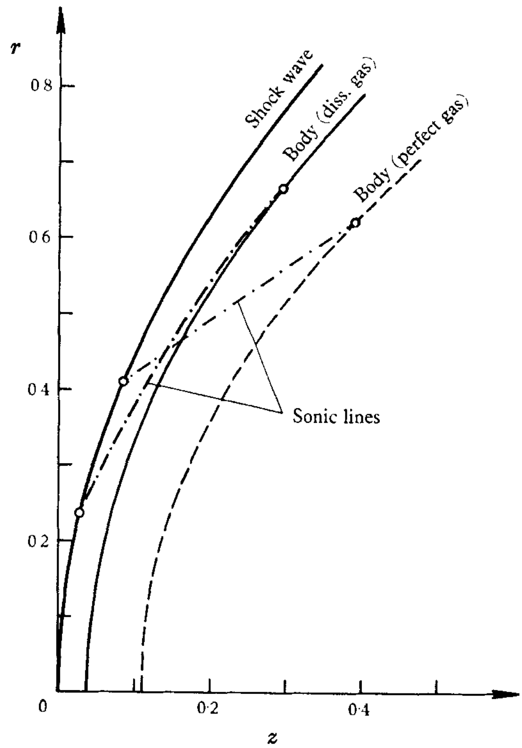

FIGURE 8. Body shape for a paraboloidal shock wave ($j = 1$, $\hat{r}^2 = 2\hat{z}$). Ideal dissociating gas, $W_\infty^2/RT_d = 2$, $\rho_\infty/\rho_d = 10^{-9}$, $T_\infty/T_d = 5\times10^{-3}$ ($M_\infty = 17{\cdot}32$). Perfect gas, $\gamma = 1{\cdot}4$, $M_\infty = 17{\cdot}32$. Subsonic and transonic region.

Results for a paraboloidal shock wave in a stream of a dissociating diatomic gas are plotted in the figures 7–9. The state equations of the ideal dissociating gas in thermodynamic equilibrium (Lighthill 1957) have been used. The constants T_d and ρ_d are a characteristic temperature and a characteristic density, respectively, of the gas. R is the gas constant per unit mass for the undissociated (molecular) gas, α the degree of dissociation. The square of the free-stream velocity has been related to RT_d, which is the dissociation energy per unit mass. Therefore $W_\infty^2/RT_d = 1$ means that the kinetic energy of the gas in the free stream is large enough to provide half the energy required to dissociate the gas completely. (Values for oxygen: $T_d = 59{,}370$ °K, $\rho_d = 150\,\text{g/cm}^3$, $RT_d = 1{\cdot}53\times10^{11}\,\text{cm}^2/\text{sec}^2$; nitrogen: $T_d = 113{,}260$ °K, $\rho_d = 130$ g/cm³, $RT_d = 3{\cdot}35\times10^{11}$ cm²/sec².) The free-stream density ρ_∞ used in the example corresponds to an altitude of about 70 km.

Figure 8 and the upper half of figure 7 show that the thickness of the shock layer is considerably smaller for the dissociating gas than for the perfect gas with constant specific heats. However, the differences in the pressures on the body surfaces are rather small, as may be seen in the lower half of figure 7. It is interesting that the degree of dissociation on the body surface, denoted by α_b, varies only slightly, although the pressure falls simultaneously to a small fraction of its value at the nose. This is due to the small temperature change conditioned by the fact that the effective isentropic exponent γ_e is nearly 1. In figure 9 the distributions

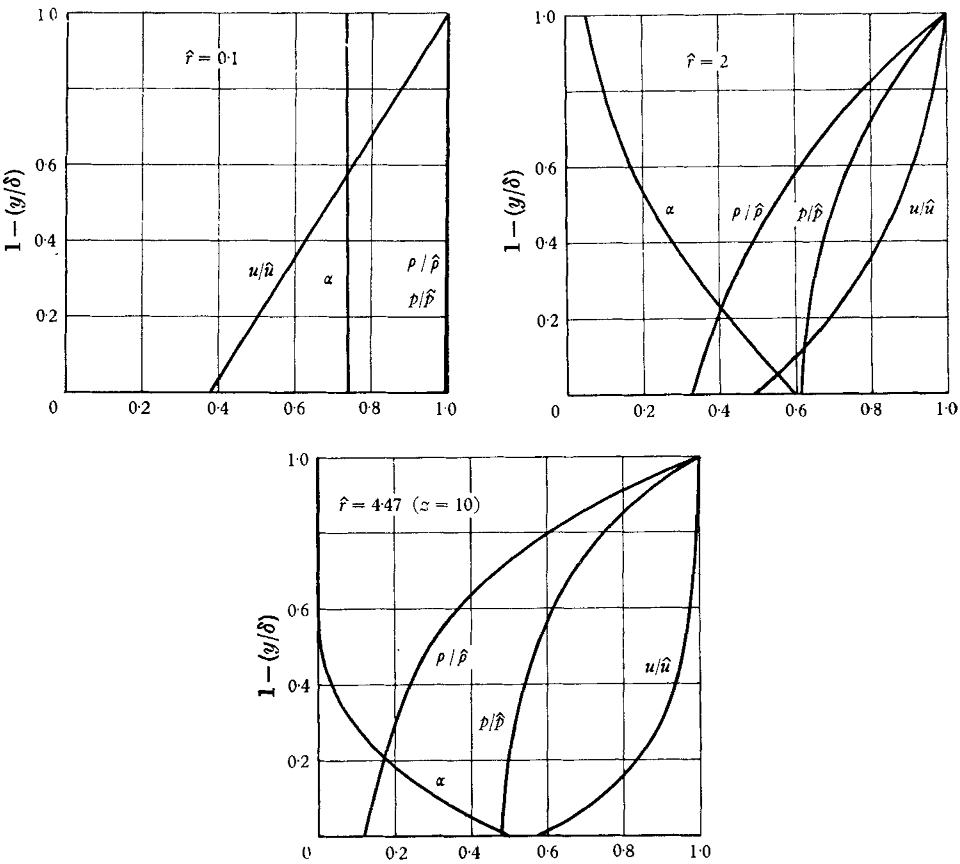

FIGURE 9. Distribution of flow quantities on three normals to a paraboloidal shock wave ($j = 1$, $\hat r^2 = 2\hat z$). Ideal dissociating gas, $W_\infty^2/RT_d = 2$, $\rho_\infty/\rho_d = 10^{-9}$, $T_\infty/T_d = 5\times 10^{-3}$ ($M_\infty = 17{\cdot}32$).

of p, ρ, u and α on normals to the shock surface are plotted. The normal $\hat r = 0{\cdot}1$ is located near the stagnation point, $\hat r = 2$ is located in the near supersonic region of the flow field, and the normal $\hat r = 4{\cdot}47$ ($\hat z = 10$) represents the flow in rather large distance from the body nose.

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REFERENCES

BROCHER, E. F. 1960 Comments on the behaviour of Sedov's blast-wave solution as $\gamma \to 1$. *J. Aero/Space Sci.* **27**, 955–6.

BUSEMANN, A. 1933 Flüssigkeits- und Gasbewegung. In *Handwörterbuch der Naturwissenschaften*, vol. IV, 2nd ed., pp. 244–79. Jena: Gustav Fischer.

CHENG, H. K. & GAITATZES, G. A. 1966 Use of the shock-layer approximation in the inverse hypersonic blunt-body problem. *AIAA J.* **4**, 406–13.

CHESTER, W. 1956 Supersonic flow past a bluff body with a detached shock. Part I, Two-dimensional body. *J. Fluid Mech.* **1**, 353–65. Part II, Axisymmetrical body. *J. Fluid Mech.* **1**, 490–6.

FREEMAN, N. C. 1956 On the theory of hypersonic flow past plane and axially symmetric bluff bodies. *J. Fluid Mech.* **1**, 366–87.

FREEMAN, N. C. 1960 A note on the explosion theory of Sedov with application to the Newtonian theory of unsteady hypersonic flow. *J. Aero/Space Sci.* **27**, 77–8 and 956.

HAYES, W. D. & PROBSTEIN, R. F. 1966 *Hypersonic Flow Theory*, vol. I, 2nd ed. New York: Academic Press.

HONDA, M. 1965 Stream-function co-ordinates in rotational flow and analysis of the flow in a shock layer. Part I. *J. Inst. Math. Applics.* **1**, 127.

HONDA, M. 1966 Stream-function co-ordinates in rotational flow and analysis of the flow in a shock layer. Part II. *J. Inst. Math. Applics.* **2**, 55.

INOUYE, M. & LOMAX, H. 1962 Comparison of experimental and numerical results for the flow of a perfect gas about blunt-nosed bodies. *NASA TN* D-1426.

LIGHTHILL, M. J. 1957 Dynamics of a dissociating gas. Part I. Equilibrium flow. *J. Fluid Mech.* **2**, 1–32.

MASLEN, S. H. 1964 Inviscid hypersonic flow past smooth symmetric bodies. *AIAA J.* **2**, 1055–61.

MIRELS, H. 1962 Hypersonic flow over slender bodies associated with power-law shocks. In *Advances in Applied Mechanics* (H. L. Dryden & Th. v. Karman, eds.), vol. 7, pp. 1–54, 317–19. New York: Academic Press.

SCHNEIDER, W. 1966 Über die Theorie dünner Hyperschall-Störschichten. DLR-Forschungsbericht 66-42. Zentralstelle für Luftfahrtdokumentation und -information, München.

SYCHEV, V. V. 1960 On the theory of hypersonic gas flow with a power-law shock wave. *Prikladnaya Matematika i Mekhanika*, **24**, 518–23, transl. in *J. Appl. Math. Mech.* **24**, 756–64.

VAN DYKE, M. D. 1958 A model of supersonic flow past blunt axisymmetric bodies, with application to Chester's solution. *J. Fluid Mech.* **3**, 515–22.

VAN DYKE, M. D. 1966 The blunt-body problem revisited. In *Fundamental Phenomena in Hypersonic Flow* (J. G. Hall, ed.), pp. 52–65. Cornell University Press.

VAN DYKE, M. D. & GORDON, H. D. 1959 Supersonic flow past a family of blunt axisymmetric bodies. *NASA Tech. Rept.* R-1.

WHITHAM, G. B. 1957 A note on the stand-off distance of the shock in high speed flow past a circular cylinder. *Comm. Pure Appl. Math.* **10**, 531–5.

YAKURA, J. K. 1962 Theory of entropy layers and nose bluntness in hypersonic flow. In *Hypersonic Flow Research* (F. R. Riddell, ed.), pp. 421–70. New York: Academic Press.